Amazing quadratic equations-2

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Why the Quadratic Formula Isn't That Great (Or Why I Dislike It)

Let us consider a simplified form of the quadratic equation, where the leading coefficient is 1:

$$x^2 + px + q = 0 \tag{1}$$

where p and q are complex numbers (they may, of course, also be real numbers).

1. Where does the Quadratic Formula come from?

The well-known quadratic formula states that if $p^2 \geq 4q$, then the solutions (possibly repeated if $p^2 = 4q$) to equation (1) are:

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \ . \tag{2}$$

This formula is not difficult to derive, and students should be encouraged to work through the derivation themselves. The key step lies in *completing the square*:

$$x^2 + px + q = 0 \iff x^2 + px + \left(\frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q,$$

which is equivalent to

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2 - 4q}{4} \ . \tag{3}$$

Here comes the major assumption: Assume that $p^2 \geq 4q$, then the above equation is equivalent to

$$\left(x + \frac{p}{2}\right)^2 = \left(\sqrt{\frac{p^2 - 4q}{4}}\right)^2 \ .$$

This leads directly to the quadratic formula (2).

2. Why shall we be cautious when using the quadratic formula (2)?

Although all textbooks state that under the assumption of $p^2 \geq 4q$, we have formula (2), students often overlook checking the condition.

Example 7: Solve the equation

$$x^2 + 2x + 5 = 0.$$

Using formula (2), we get

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2}.$$

Now, what is $\sqrt{-16}$? One might argue, based on the exponent rule:

$$(ab)^r = a^r b^r \tag{5}$$

that

$$\sqrt{-16} = \sqrt{16} \times \sqrt{-1} = 4i. \tag{6}$$

So we arrive at the solutions:

$$x = -1 + 2i \;\; or \; x = -1 - 2i.$$

And these answers are correct!

So, what's the problem?

Even if we arrive at the correct solution, the method may be flawed. Specifically, identity (5) does **not** hold when a and b are negative and r is a fractional exponent (like $1/2$). Thus, expressions such as $\sqrt{-16}\; or \;\sqrt[3]{-1}$ can be ambiguous or misleading unless we define them very carefully.

In fact, we do **not** recommend using expressions like $\sqrt{-16}$ unless you treat identity (6) as a **definition**, not a consequence of general exponent rules.

A Better Approach Without Using $\sqrt{-16}$

So, how should we solve the equation?

Recall the derivation of the quadratic formula. While formula (2) holds only under certain conditions, the *completing-the-square* form:

$$\left(x+\frac{p}{2}\right)^2 = \frac{p^2-4q}{4}\,. \tag{3}$$

is always valid for any complex numbers p and q.

Solution to Example 7 (using completing the square):

$$x^2+2x+5=0 \Leftrightarrow (x+1)^2=-4 \Leftrightarrow (x+1)^2=(2i)^2.$$

Therefore, the solutions are: $x=-1+2i, or\ x=-1-2i.$

In this approach, we avoid ambiguous expressions like $\sqrt{-16}$ and use a mathematically sound method that works in all cases.

A Common Mistake: Solving $x^2=r$

Let us return to the simplest quadratic equation:

$$x^2=r,$$

where r is a complex number.

A common mistake is to write:

$$x=\sqrt{r} \quad or \quad x=-\sqrt{r}$$

This is only acceptable if $r \geq 0$. Even then, one must ask: *why are these the only two solutions?*

If $r<0$, or r is a general complex number, the notation $\sqrt{r}$ becomes ambiguous. For instance, if $r=i$, what exactly is $\sqrt{i}$? How do we know it is a complex number, and how many such square roots are there?

In the **Everyone Math** curriculum, we discourage students from using $\sqrt{r}$ unless r is nonnegative. Instead, we teach how to solve the equation $x^2=r$ for complex r in Solving equation series-5 of this note. Combining that method with formula (3), we arrive at a powerful and elegant conclusion:

Every quadratic equation of the form $x^2 + px + q = 0$ has a solution in the complex numbers.

This is essentially the **Fundamental Theorem of Algebra** for quadratic equations.

Example 8: Solve the equation

$$x^2 + 2ix + 5 = 0.$$

Solution:

$$x^2 + 2ix + 5 = 0 \Leftrightarrow (x + i)^2 = -6 \Leftrightarrow (x + i)^2 = \left(\sqrt{6}i\right)^2.$$

Thus, we obtain: $x = (-1 + \sqrt{6})i, or\ x = (-1 - \sqrt{6})i.$

Summary

By completing the square, any quadratic equation of the form

$$\boldsymbol{x^2 + px + q = 0}$$

can be rewritten as:

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2 - 4q}{4}.$$

This form is valid for all complex numbers p and q, and it always leads to a solution in the complex number field. This illustrates the **Fundamental Theorem of Algebra** in the case of degree-2 polynomials and highlights a more conceptually sound alternative to blindly applying the quadratic formula.